Logic and Reasoning: Difference between implication and entailment (logical implication)?

I was asked about the differences between the two symbols by a student.

\[ p \implies q \quad (1) \]

and

\[ p \models q \quad (2) \]

The first one is a material implication, or whatever people call it, which expresses a binary truth function in propositional logic. So, in propositional logic, we can build a truth table to see if (1) holds. This definition is not often desired because as we know q can be anything when p is false, and this is still a tautology. Some claim that this is paradoxical (cf. Wikipedia).

A simple way to understand: (1) \( p \implies q \) is merely a shorthand (logical equivalence) for \( \neg p \lor q \), which means that they can be used interchangeably. A material implication is truth-preserving, and that is what it does.

In predicate logic, I understand (1) as follows. The two sets are compared in terms of the extensions of the two sets, e.g.,

\[ A = \{ \text{forall } x: x > 0 \} \]

\[ B = \{ \text{forall } x: x > -1 \} \]

then we say \( A \implies B \).

People claim that (1) introduces paradox (see wikipedia) because this sort of implication does not exclude apparent contradictions, e.g.,

"4 is odd \( \implies \) 4 is even" \( (3) \)

is true. So, it is said that (1) does not correspond to "if p then q", but rather "not p or q".

However, a paper by Ceniza in 1988 suggested that (3) is not a paradox because from the material implication (1) we can in fact get the following formula:

\[ q \lor p \implies q \quad (4) \]

So (3) is a tautology because the material implication has put the consequent (that is, q) implicitly in the antecedent, which has nothing to do with the false antecedent \( p \) here as q implies/entails itself here! I make no comment on this paper, and include this for the sake of completeness only.

(2) is also called entailment, logical implication, semantic implication, logical consequence, etc. because it relates to the model theory saying that every model/interpretation of p is also a model of q. This is somewhat stronger than (1) from the semantic point of view. An explanation follows.
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Entailment like "p1, p2, ... pk |= q" states that whenever the valuation makes p1, p2, ..., pk true, then it must make q true. That is, the conclusion follows deductively from the premises. This does not mean that the conclusion is true, as the premises can be false [cf. Tarski]. Entailment means it is impossible for this case to occur: the premises are true and the consequence is false.

In deductive systems, the deduction theorem applies to most logics (but not all): if p|= q then EMPTY |= (p => q), and vice versa. So another understanding in these systems can be: an entailment like p|= q states that (p => q) is always true, or a tautology.

Note that implication is in the logic itself (it is normally defined as an operator/connective of the logic), while entailment is in the meta-logic (a language used to describe some properties of a logic).

Added notes for beginners:

The symbol called proof or consequence |- is obvious different from |=, the logical consequence. A proof manipulates the structure (syntax) of formulas, and tries to reach another formula by referring to a set of rules. If indeed such a proof is found, we say from the premises we can conclude the consequent.

Logical consequence, or entailment instead looks at the model/valuation of formulas in the premises and that of the conclusion.

In a nutshell, consequence (denoted by |-) is syntactic, while entailment (denoted by |=) is related to semantics, thus semantic.